Heat kernel regularization of the effective action for stochastic reaction-diffusion equations

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The presence of fluctuations and nonlinear interactions can lead to scale dependence in the parameters appearing in stochastic differential equations. Stochastic dynamics can be formulated in terms of functional integrals. In this paper we apply the heat kernel method to study the short distance renormalizability of a stochastic (polynomial) reaction-diffusion equation with real additive noise. We calculate the one-loop *effective action* and its ultraviolet scale dependent divergences. We show that for white noise a polynomial reaction-diffusion equation is one-loop *finite* in $d=0$ and $d=1$, and is one-loop renormalizable in $d=2$ and $d=3$ space dimensions. We obtain the one-loop renormalization group equations and find they run with scale only in $d=2$.

I. INTRODUCTION

Many examples abound where particular spacetime distributions of matter are selected over a wide variety of seemingly possible choices. In many of these cases, these patterns are well described, and their temporal evolution accurately modeled, by nonlinear partial differential equations subject to noise, or equivalently, by stochastic partial differential equations (SPDEs). Concrete examples can be found in the domains of pattern formation, chemical chaos, biological morphogenesis, and flame-front propagation, just to name a few [1–3]. As argued in Ref. [4] the *effective potential* is a superb tool for studying the onset of pattern formation (i.e., symmetry breaking) about the static and spatially homogeneous solutions of the SPDE. This quantity takes into account both interactions and fluctuations to a given number of loops. (The loop expansion is a controlled expansion in the amplitude of the fluctuations.) To go beyond the study of symmetry breaking and static homogeneous configurations, it is crucial to be able to include and account for time development and spatial inhomogeneities. Ultimately, we are interested in studying the late time behavior of the solutions of SPDEs; we would like to know if there is an asymptotic steady state, or state of equilibrium, as this impinges on the late time behavior of the emergent pattern. The effective potential does not yield this information, and one must turn to the *effective action*. The effective action contains all the dynamical solutions of the SPDE and their asymptotic behavior. It obeys a variational principle wherein its first variation yields the SPDE. Its exact calculation, however, is very difficult. Along the way, one must first ''regularize'' its short distance divergences that are dimension dependent. These must be identified, isolated, regulated, and, if possible, renormalized by suitable redefinitions of the bare parameters appearing in the SPDE. The study of these dimension dependent divergences lends itself to a direct calculation of the renormalization group equations (RGEs) that encode the scale dependence (or ''running'') of these parameters. The RGE fixed points provide information about the asymptotic states. For these and other reasons, the divergent structure of the effective action warrants a detailed analysis in its own right.

In this paper we study the stochastic reaction-diffusion (RD) equation by means of a functional integral representation that will be used to calculate its one-loop effective action. In earlier work we calculated and analyzed the *effective potential* for the RD equation (the effective action evaluated for special field configurations that are time independent) [5]. However, the effective potential does not provide information about the dynamics of the system or wavefunction renormalization. The effective potential can only signal the static homogeneous states around which one can study the onset of pattern formation (i.e., symmetry breaking), but does not indicate *which* such state (if any) is dynamically accessible or most likely. To go beyond the limitations inherent in an effective potential analysis, and in order to judge the importance of the noise-induced symmetry broken ground state configuration, one should explore the space of dynamical (spatially inhomogeneous) solutions of the system. For instance, are there solutions that indicate the system is ''thermalizing,'' or can we find steady state solutions? An important step in this direction is the analysis of the effective action undertaken in this paper. In this calculation we encounter short distance divergences that need to be regularized. We have chosen to carry out this regularization procedure by means of (generalized) heat kernel techniques. To do so, we will be required to calculate the first (integrated) Seeley-DeWitt coefficients. In addition to the physical RD field itself, we must also calculate the contribution from the ghost field, a necessary ingredient in this formalism [4].

After the regularization step, we turn to the one-loop renormalization of the effective action. We find that for additive white noise RD systems are one-loop finite in $d=0$ and $d=1$ space dimensions, and (at least) one-loop renormalizable in $d=2$ and $d=3$ space dimensions for polyno-

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mial reaction kinetics. There is no wavefunction renormalization in any of these dimensions, irrespective of the degree of the reaction polynomial [6]. Moreover, the ultraviolet renormalizability requires including a bare tadpole λ_0, or constant term, in the equation of motion.

Application of heat kernel methods to stochastic field theories far from equilibrium may not be familiar, so a few comments regarding them may be useful. These techniques have been used primarily for calculating effective actions and the one-loop physics in quantum field theories (QFTs) and in curved space quantum field theory [7–12]. In the quantum domain, one is interested in computing quantities such as the *effective action* and the *effective potential*, which provide crucial information regarding the structure of the underlying theory at different length and time scales and are important in assessing the theory's renormalizability (or lack thereof), the determination of the running of couplings and parameters, patterns of spontaneous and dynamical symmetry breaking, and the structure of short distance (ultraviolet) and long distance (infrared) divergences [13–17]. Moreover, for renormalizable theories, the computation of the effective action (actually, only its divergent part is needed) can be used to extract the RGEs that govern the scale dependence of the couplings and parameters appearing in the theory [18–20]. Though perhaps better known in the context of these fields, these same techniques can be generalized and applied to reveal the corresponding one-loop physics associated with stochastic dynamic phenomena and to systems subject to fluctuations.

As a key technical step, we need to obtain the Green functions for certain higher-order differential operators that are neither elliptic nor hyperbolic. (The operators treated here are second-order in physical time derivatives and fourth-order in space derivatives.) We set up a generalization of the Schwinger proper time asymptotic expansion to compute these Green functions and obtain explicit expressions for the first Seeley-DeWitt coefficients associated with these operators. We carry this out employing the minimal representation in which only physical and ghost field variables appear in the action [4].

At this stage we emphasize the following: We start from a given ''classical'' SPDE, together with a specification of the type of noise and its correlations. Our aim is to *map* this SPDE to an equivalent generating functional and effective action, at which point QFT techniques can be used. We emphasize that our approach and use of field theory is rather different from that initiated some years ago, whose aim is to *derive* the SPDE and the noise correlations from microphysics, by mapping a classical master equation to ''second-quantized'' variables, and then finally to an action principle [21–23].

We conclude with a summary and discussion of this work, and prospects for further development.

II. STOCHASTIC FIELD THEORY FOR REACTION-DIFFUSION EQUATIONS

In this paper we demonstrate that heat kernel techniques can be used to compute the ultraviolet divergent terms aris-

ing in the one-loop effective action associated with field theory formulations of SPDEs. In particular, we apply these methods to the class of single component reaction-diffusion equations

$$D\phi(x) = V[\phi(x)] + \eta(x), \tag{1}$$

where

$$D = \frac{\partial}{\partial t} - \nu \nabla^2, \tag{2}$$

and

$$V[\phi] = \sum_{j=0}^{N} \frac{\lambda_j}{j!} \phi^j(x). \tag{3}$$

Here $\eta(x)$ is a noise term with normalized probability distribution given by $P[\eta]$. We employ the shorthand notation $x = (\vec{x}, t)$. Furthermore, $V[\phi]$ is a polynomial of degree N in the concentration (or field variable ϕ) and the λ_j's are a set of reaction rates. For convenience we have placed the decay (or ''mass'' term) into the potential and have treated it as another coupling: $\lambda_1 \equiv -m^2$. It must be noted that this equation has the form of a purely dissipative system and has a bona-fide potential energy term $V[\phi]$ [15]. Thus, it makes sense to calculate an *effective potential* for constant field configurations, as well as an effective action for inhomogeneous fields. Both the effective action and the effective potential are derived by means of a field theory for this SPDE.

Before continuing, we should point out that there are reasons to believe that as a *phenomenological* equation, the RD being considered here (and others structurally similar to it), might not be adequate to describe certain two-body annihilation processes, or pair reaction kinetics, since recent derivations based on master equations show that the SPDE in question should actually be *complex* with *imaginary* noise (leading to negative noise correlations) [6,24,25]. On the other hand, for processes involving particle clustering, these same derivations yield real stochastic equations and noise, as well as positive noise correlations [26]. Of course, there are many situations in which a microscopic derivation of the SPDE is entirely out of the question, either because explicit knowledge of the microscopic details is lacking and/or because the random fluctuations owe to uncontrollable contingencies. In these situations the benefit of adopting a phenomenological strategy should be self-evident. Finally, the application of this equation need not be restricted to just chemical diffusion [27].

For homogeneous and static concentrations it is sufficient to study the effective potential [5]. In this paper we complement that analysis by making use of the effective action to consider inhomogeneous and time dependent field configurations. In the minimalist approach (see Ref. [4]) one starts with the normalized generating functional $Z[J]$ encoding the stochastic dynamics described by the RD equation (1). This involves the RD scalar field ϕ plus the Jacobian determinant (denoted here by $\mathcal{J}$) and its adjoint ($\mathcal{J}^\dagger$) (these determinants arise from a change of variables). The generating functional (partition function) is given by [4,15]

$$Z[J] = \frac{\int [d\phi] \exp\left\{\left[-S[\phi] + \int d^n x\, J(x)\phi(x)\right]\Big/\mathcal{A}\right\} \sqrt{\mathcal{J}\mathcal{J}^\dagger}}{\int [d\phi]\exp\{-S[\phi]/\mathcal{A}\} \sqrt{\mathcal{J}\mathcal{J}^\dagger}}, \tag{4}$$

where the ''classical'' action is [valid for Gaussian noise: $G_\eta(x,y) = \langle \eta(x)\eta(y) \rangle = \mathcal{A}\, g_2(x,y)$]

$$S[\phi] = \tfrac{1}{2} \int d^n x_1 \int d^n x_2\, (D\phi(x_1)$$
$$- V[\phi(x_1)])\, g_2^{-1}(x_1,x_2)\, (D\phi(x_2) - V[\phi(x_2)]), \tag{5}$$

with $n = d+1$ and d the number of space dimensions (we will keep d as a free parameter throughout this paper). For a general SPDE there may be nonvanishing contributions from the ''ghost'' fields (Jacobian determinants). We follow here the discussion of Ref. [4] to separate the noise two-point function into the product of a *shape* $g_2(x,y)$ and a constant amplitude $\mathcal{A}$. Irrespective of how we decide to normalize the shape, the constant amplitude $\mathcal{A}$ is always the *loop-counting* parameter of the perturbation expansion [4]. A loop-counting parameter is very useful in organizing such a perturbative expansion. Moreover, any symmetry that is present in the classical action (5) is preserved at each order in the loop expansion since the loop-counting parameter multiplies the entire action (and the source term J) in Eq. (4). One of the advantages of the minimal representation is that it leads to this natural identification of the noise amplitude [4,15].

We introduce the generating functional for connected correlation functions $W[J]$ and its Legendre transform, the effective action $\Gamma[\bar\phi]$ [15] (note the explicit factor of the noise amplitude $\mathcal{A}$)

$$W[J] = +\mathcal{A} \log Z[J], \tag{6a}$$

$$\Gamma[\bar\phi] = -W[J] + \int d^n x\, J(x)\{\bar\phi[J](x) - \bar\phi[0](x)\}, \tag{6b}$$

with

$$\bar\phi[J] = \frac{\delta W[J]}{\delta J}. \tag{7}$$

The barred fields $\bar\phi[J]$ and $\bar\phi[0]$ are the solutions of the equations of motion

$$\left(\frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi}\right)\Bigg|_{\bar\phi[J]} = J(x), \quad \text{and} \quad \left(\frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi}\right)\Bigg|_{\bar\phi[0]} = 0, \tag{8}$$

respectively. It is usually assumed that the former equation has a unique solution $\bar\phi[J]$ (at least for small J), and that for vanishing source ($J=0$) the unique solution is the vanishing

mean field $\bar\phi[0] = 0$, i.e., $\langle \phi(\vec{x},t) \rangle_{J=0} = 0$, where the angular brackets denote the stochastic average with respect to the noise probability distribution $\mathcal{P}[\eta]$. This is valid for a symmetric ground state. We next expand the action (5) about this mean field up to quadratic order in the stochastic fluctuation $\delta\phi = \phi - \bar\phi$. We can carry out a perturbation expansion in the small parameter $\mathcal{A}$ and compute the one-loop effective action to obtain [13][1]

$$\Gamma[\phi] = S[\phi] - S[0] + \frac{\mathcal{A}}{2}(\text{Tr}\log S^{(2)}_{\text{field}}[\phi]$$
$$- \text{Tr}\log S^{(2)}_{\text{field}}[0] - \log \mathcal{J}[\phi] - \log \mathcal{J}^\dagger[\phi]$$
$$+ \log \mathcal{J}[0] + \log \mathcal{J}^\dagger[0]) + O(\mathcal{A}^2)$$
$$= S[\phi] - S[0] + \Gamma^{(1-\text{loop})}[\phi] + O(\mathcal{A}^2)$$
$$= S[\phi] - S[0] + \Gamma^{(1-\text{loop})}_\epsilon[\phi] + \Gamma^{(1-\text{loop})}_{\text{finite}}[\phi] + O(\mathcal{A}^2), \tag{9}$$

where the matrix elements of the Jacobi field operator $S^{(2)}_{\text{field}}[\phi]$ are

$$\langle x_1 | S^{(2)}_{\text{field}}[\phi] | x_2 \rangle = S^{(2)}_{\text{field}}(\phi, x_1, x_2) = \frac{\delta^2 S[\phi]}{\delta\phi(x_1)\,\delta\phi(x_2)}. \tag{10}$$

We have anticipated the appearance of divergences in the one-loop contribution to the effective action, arising from both the physical and ghost fields, and have supplied it with a cut-off ϵ.

The notation $S[0]$ is actually shorthand for $S[\phi[J=0]]$, and for a symmetric ground state one typically has $\phi[0] = 0$ and $S[0] = 0$, unless there is a ''tadpole'' in the classical action. In fact, when looking for mean field solutions of the zero-loop equation of motion, we will find it convenient to consider a nonvanishing value of the mean field $\phi[0] = v_0 \neq 0$ and will study fluctuations about this mean value. The terms involving $S[0]$ and $S^{(2)}_{\text{field}}[0]$ appear due to the normalization factor in Eq. (4). The notation ''Tr'' stands for the trace and indicates that we are to take the (time and space) coincidence limits $x_2 \to x$ and $x_1 \to x$, followed by an integration over the common limit x. The one-loop effective action will contain divergent terms and it is precisely these terms we wish to isolate and compute. We have collected all such

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[1]We now drop the overbar on ϕ with the understanding that this stands for the conjugate field of J and not the field appearing in the classical (zero-loop) action.

divergences into the expression $\Gamma_\epsilon^{(1-\text{loop})}$ and we regulate them by means of a cut-off ϵ. The finite terms are collectively represented by $\Gamma_{\text{finite}}^{(1-\text{loop})}$. There may also be higher-loop contributions, denoted by $O(\mathcal{A}^2)$, whenever we need to emphasize them explicitly. Although these latter contributions are important for constructing the full effective action, that calculation is beyond the scope of the present paper.

In order to compute the one-loop effective action we need to obtain $S_{\text{field}}^{(2)}[\phi]$. This Jacobi field operator is diagonal in coordinate space. For the purpose of this calculation and in the interest of simplicity, we consider the case of white noise. (Colored noise can be dealt with, but it brings in time and space derivatives of the shape function, which complicate the heat kernel analysis.) For white noise we have $\langle \eta(x) \eta(y) \rangle = 2 D_0\, \delta^n(x,y)$ and therefore we can write

$$G_\eta(x,y) = 2 D_0\, \delta^n(x,y), \Rightarrow \mathcal{A} = 2 D_0,$$

$$g_2(x,y) = \delta^n(x,y), \quad \text{and} \quad g_2^{-1}(x,y) = \delta^n(x,y),$$

(11)

which fixes the noise normalization.

The Jacobi field operator corresponding to the RD equation is easy to calculate starting from the classical action. We simplify notation and write the zero-noise action as

$$S[\phi] = \tfrac{1}{2} \int d^n x\, (D\phi - V[\phi])^2$$

$$= \tfrac{1}{2} \int d^n x\, (\partial_t \phi - \nu\nabla^2\phi - V[\phi])^2. \quad (12)$$

The Jacobi operator for the physical field, $S_{\text{field}}^{(2)}(\phi,x_1,x_2)$, is given by

$$S_{\text{field}}^{(2)}(\phi,x_1,x_2) \equiv [(-\partial_t - \nu\nabla^2 - V'[\phi(x_1)])(\partial_t - \nu\nabla^2$$
$$- V'[\phi(x_1)]) - V''[\phi(x_1)](D\phi(x_1)$$
$$- V[\phi(x_1)])]\delta^n(x_1,x_2), \quad (13)$$

where $V'[\phi] = dV[\phi]/d\phi$ and $V''[\phi] = d^2V[\phi]/d\phi^2$.

For the ghost field the Jacobi operator is given by [4]

$$S_{\text{ghost}}^{(2)}(\phi,x_1,x_2) \equiv [(-\partial_t - \nu\nabla^2 - V'[\phi(x_1)])(\partial_t - \nu\nabla^2$$
$$- V'[\phi(x_1)])]\delta^n(x_1,x_2), \quad (14)$$

and its determinant can be written as [4]

$$\det[S_{\text{ghost}}^{(2)}(\phi,x_1,x_2)] = \mathcal{J}[\phi]\, \mathcal{J}^\dagger[\phi]. \quad (15)$$

In order to carry out the perturbation expansion we also need to consider the "free" case defined by the limit $V[\phi] \to 0$

$$S_{\text{free}}^{(2)}(x_1,x_2) = [(-\partial_t - \nu\nabla^2)(\partial_t - \nu\nabla^2)]\,\delta^n(x_1,x_2)$$

$$= [-\partial_t^2 + (\nu\nabla^2)^2]\,\delta^n(x_1,x_2). \quad (16)$$

Free physical fields have the same Jacobi operator as free ghost fields, so that as $V[\phi] \to 0$, the physical and ghost field contributions to the effective action cancel. We now look

ahead a little: as the Jacobi operator $S_{\text{free}}^{(2)}(x_1,x_2)$ contains fourth order space derivatives (*bi-harmonic operator*), rather than second order derivatives, the behavior of the DeWitt-Schwinger expansion [7–11] is *qualitatively* different in that it includes *fractional* powers of the Schwinger proper time parameter.

We now calculate the mean field v_0 (i.e., the background field) by studying the solutions of the classical equation of motion, which is given (for arbitrary source J) by

$$\left(\frac{\delta S}{\delta \phi}\right)\Bigg|_{\phi[J]} = (-\partial_t - \nu\nabla^2 - V'[\phi(x)])(D\phi(x) - V[\phi(x)])$$

$$= J(x). \quad (17)$$

If the source vanishes and the mean field is homogeneous and static, we have

$$V'[v_0]\, V[v_0] = 0, \quad (18)$$

which always has at least one real solution [5].

In order to calculate the one-loop effective action for RDs, one must include the contribution from the "ghost" fields. These "ghost" Jacobians are given by [5]

$$\mathcal{J} = \det\left(D - \frac{\delta V}{\delta \phi}\right), \quad \text{and} \quad \mathcal{J}^\dagger = \det\left(D^\dagger - \frac{\delta V^\dagger}{\delta \phi}\right). \quad (19)$$

We can now complete the formal calculation of the one-loop effective action. We have [4]

$$\Gamma[\phi] = S[\phi] - S[v_0] + \frac{\mathcal{A}}{2}\Bigg[\text{Tr} \log S_{\text{field}}^{(2)}[\phi]$$

$$- \text{Tr} \log\left(D^\dagger - \frac{\delta V^\dagger}{\delta \phi}\right)\left(D - \frac{\delta V}{\delta \phi}\right)$$

$$- (\phi \to v_0)\Bigg] + O(\mathcal{A}^2), \quad (20)$$

so the one-loop effective action receives one contribution from the physical field

$$\Gamma_{\text{field}}^{(1-\text{loop})} = \frac{\mathcal{A}}{2}(\text{Tr} \log S_{\text{field}}^{(2)}[\phi] - \text{Tr} \log S_{\text{field}}^{(2)}[v_0]), \quad (21)$$

and a contribution from the ghost field

$$\Gamma_{\text{ghost}}^{(1-\text{loop})} = -\frac{\mathcal{A}}{2}\Bigg[\text{Tr} \log\left(D^\dagger - \frac{\delta V^\dagger}{\delta \phi}\right)\left(D - \frac{\delta V}{\delta \phi}\right)_\phi$$

$$- \text{Tr} \log\left(D^\dagger - \frac{\delta V^\dagger}{\delta \phi}\right)\left(D - \frac{\delta V}{\delta \phi}\right)_{v_0}\Bigg]. \quad (22)$$

We will soon see that individually, each contribution has complicated Seeley-DeWitt coefficients, but when taken together, the physical plus ghost sectors yield simple *net* Seeley-DeWitt coefficients.

III. COMPUTING THE ONE-LOOP EFFECTIVE ACTION

In this section we construct the regulated expression $\Gamma_\epsilon^{(1-\text{loop})}[\phi]$ for the RD equation. We follow closely the DeWitt-Schwinger (DS) proper time formalism to analyze the ultraviolet divergences [7–11,14–16]. (We have striven to keep this section self-contained.)

In this formalism the integral representation for $\Gamma^{(1-\text{loop})}[\phi]$, Eq. (9), involves a fictitious "time" parameter s (denoted as Schwinger proper time). To this end, we define the following function, where A is any operator

$$g_\alpha(A) \equiv \int_0^{+\infty} ds \, s^{\alpha-1} e^{-sA} = A^{-\alpha} \Gamma(\alpha), \quad (23)$$

with $\Gamma(\alpha)$ the Gamma function. We consider the limit $\alpha \rightarrow 0$

$$g_\alpha(A) \rightarrow \frac{1}{\alpha} - \gamma - \log A, \quad (24)$$

where $\gamma = 0.577\ldots$, is Euler's constant. Although this integral is divergent, the difference of *two* such integrals is finite and well defined

$$\lim_{\alpha \to 0} [g_\alpha(B) - g_\alpha(A)] = \log A - \log B$$

$$= -\int_0^{+\infty} \frac{ds}{s} (e^{-sA} - e^{-sB}), \quad (25)$$

and comparing with Eq. (9), the desired proper time integral for the one-loop effective action is given by

$$\Gamma^{(1-\text{loop})}[\phi] = \Gamma_{\text{field}}^{(1-\text{loop})}[\phi] + \Gamma_{\text{ghost}}^{(1-\text{loop})}[\phi] \quad (26)$$

$$= -\frac{\mathcal{A}}{2} \int_0^{+\infty} \frac{ds}{s} \text{Tr}(e^{-sH_{\text{field}}} - e^{-s[H_0]_{\text{field}}})$$

$$+ \frac{\mathcal{A}}{2} \int_0^{+\infty} \frac{ds}{s} \text{Tr}(e^{-sH_{\text{ghost}}} - e^{-s[H_0]_{\text{ghost}}}), \quad (27)$$

where the "Hamiltonians" in the exponentials are the Jacobi operators

$$H_{\text{field}} = S_{\text{field}}^{(2)}[\phi[J]] = (D^\dagger - V')(D - V') - V''(D\phi - V), \quad (28a)$$

$$[H_0]_{\text{field}} = S_{\text{field}}^{(2)}[\phi[0]], \quad (28b)$$

$$H_{\text{ghost}} = S_{\text{ghost}}^{(2)}[\phi[J]] = (D^\dagger - V')(D - V'), \quad (28c)$$

$$[H_0]_{\text{ghost}} = S_{\text{ghost}}^{(2)}[\phi[0]], \quad (28d)$$

$$H_{\text{free}} = D^\dagger D, \quad (28e)$$

as can be seen by comparing Eq. (26) with Eqs. (9), (13), (14), and (16). To proceed with the calculation, we need an explicit form for the operators e^{-sH} (that is, for $e^{-sH_{\text{field}}}$, $e^{-s[H_0]_{\text{field}}}$, $e^{-sH_{\text{ghost}}}$, and $e^{-s[H_0]_{\text{ghost}}}$), or rather, their matrix elements, so that we can take the indicated traces. To solve

for them, we note that e^{-sH} is the exact solution of the following operator differential equation

$$He^{-sH} = -\frac{\partial}{\partial s} e^{-sH}. \quad (29)$$

If one takes matrix elements in the spacetime coordinate basis $|x\rangle \equiv |\vec{x}, t\rangle$, inserts a complete set of states, and makes use of the diagonality of H in the coordinate basis [note that $S_{\text{field}}^{(2)}[\phi]$ is proportional to $\delta^n(x_1, x_2)$], we obtain

$$H(x) \langle x | e^{-sH} | x' \rangle = \int dy \langle x | H | y \rangle \langle y | e^{-sH} | x' \rangle$$

$$= \langle x | H e^{-sH} | x' \rangle$$

$$= -\frac{\partial}{\partial s} \langle x | e^{-sH} | x' \rangle, \quad (30)$$

or equivalently

$$H(x) G(x, x'|s) = -\frac{\partial}{\partial s} G(x, x'|s), \quad \text{with}$$

$$G(x, x'|s) \equiv \langle x | e^{-sH} | x' \rangle. \quad (31)$$

This latter equation defines the Green function $G(x, x'|s)$ in terms of the matrix element of the operator e^{-sH} in the coordinate representation. These steps can be repeated for the other Hamiltonians. Fortunately, for the purposes of the present work, it is not necessary to solve this equation exactly (for either H_{field} or H_{ghost}), as we are interested in the short distance divergent part of the one-loop effective action. What we will do instead is solve the "heat" equations (29), (30), and (31) adiabatically by expanding in small positive fractional powers of the proper time variable s, which is where all the ultraviolet divergences are to be found (different techniques are required if one is interested in the infrared limit). Nevertheless, to get "off the ground" it will be most useful to have the exact solution to Eq. (30) in the free limit ($V[\phi] \rightarrow 0$). We now turn to this task, which entails solving exactly Eq. (29) with H_{free}.

Since Eq. (30) looks like a heat equation in a $n+1$ dimensional spacetime (parabolic equation in the proper time variable), we know how to solve it together with specified boundary and/or initial conditions. In the free field limit ($G \rightarrow G_{\text{free}}$), we must solve the following equation

$$\left[-\partial_t^2 + (\nu \nabla^2)^2 + \frac{\partial}{\partial s} \right] G_{\text{free}}(x, x'|s) = \delta^n(x - x') \, \delta(s), \quad (32)$$

subject to the boundary (initial) condition $G_{\text{free}}(x, x'|0) = \delta^n(x - x')$. Strictly speaking, the Green function depends on both arguments x and x', but due to the translational invariance of the dynamical equation, we have $G(x, x'|s) = G(x - x'|s) = G(\vec{x} - \vec{x}', t - t'|s)$ and it suffices to treat it as a function of one spacetime coordinate. We can always restore its dependence on both spacetime arguments at any time.

The formal solution for G_{free}, expanded to fourth order in $x-x'$ (along the diagonal), is given by

$$G_{\text{free}}(x,x'|s) = A_d \exp\left[-\frac{(t-t')^2}{4s}\right] s^{-1/2-d/4}$$

$$\times \left[1 - \frac{C_{d,1}}{2d} \frac{|\vec{x}-\vec{x}'|^2}{\nu\sqrt{s}} \right.$$

$$+ \frac{C_{d,2}}{8d(d+2)} \frac{(|\vec{x}-\vec{x}'|^2)^2}{\nu^2 s}$$

$$\left. + O\left(\frac{(|\vec{x}-\vec{x}'|^2)^3}{s^{3/2}}\right) \right], \qquad (33)$$

where

$$A_d = \left(\frac{1}{4\pi}\right)^{1/2} \frac{\pi^{d/2}\,\Gamma\left(\dfrac{d}{4}\right)}{2(2\pi)^d\,\Gamma\left(\dfrac{d}{2}\right)} \left(\frac{1}{\nu^2}\right)^{d/4}, \quad \text{and}$$

$$C_{d,n} = \frac{\Gamma\left(\dfrac{d+2n}{4}\right)}{\Gamma\left(\dfrac{d}{4}\right)}. \qquad (34)$$

Details of the calculation leading to the final expression for $G^{(\text{free})}(x,x'|s)$ are given in Appendix A.

We also point out that for static and homogeneous background fields (v_0) the computation of $G_{\text{field}}[v_0]$ and $G_{\text{ghost}}[v_0]$ is not much more complicated than that for G_{free}. Details are presented in Appendix B. These static and homogeneous calculations allow one to compare with the effective potential formalism of Ref. [5], and serve as a check on the current effective action calculation.

We adopt the following ansatz to perturbatively solve the heat equations (30) for small s (adiabatic approximation)

$$G_{\text{field}}(x,x'|s) = G_{\text{free}}(x,x'|s) f_{\text{field}}(x,x'|s), \qquad (35a)$$

$$G_{\text{ghost}}(x,x'|s) = G_{\text{free}}(x,x'|s) f_{\text{ghost}}(x,x'|s), \qquad (35b)$$

where

$$f_{\text{field}}(x,x'|s) = \sum_{l=0}^{+\infty} s^{l/2} b_{l/2}(x,x')$$

$$= b_0 + s^{1/2} b_{1/2} + s\, b_1 + O(s^{3/2}), \qquad (36a)$$

$$f_{\text{ghost}}(x,x'|s) = \sum_{l=0}^{+\infty} s^{l/2} a_{l/2}(x,x')$$

$$= a_0 + s^{1/2} a_{1/2} + s\, a_1 + O(s^{3/2}), \qquad (36b)$$

are asymptotic series in half-integer powers of the proper time with coefficient functions $b_{l/2}$ and $a_{l/2}$ (called "Seeley–DeWitt" coefficients). Note that we have had to consider fractional powers in this small s expansion. (By considering simple cases it is easy to convince oneself that for a differ-

ential operator of order n the "heat kernel" expansion should start with an overall factor proportional to $s^{-d/n}$ and then contain subdominant terms that are integer powers of $s^{2/n}$.)

In principle, these coefficients can be calculated to arbitrarily high order by solving recursion relations obtained by substituting Eqs. (35a) and (35b) into Eq. (30) for H_{field} and H_{ghost}, respectively. The boundary (initial) conditions $G_{\text{field}}(x,x'|0) = G_{\text{ghost}}(x,x'|0) = \delta^n(x,x')$ imply that $b_0 = 1$ and $a_0 = 1$. These coefficients start the Seeley-DeWitt hierarchy and allow for a complete determination of the Seeley-DeWitt coefficients appearing in Eqs. (36a)–(36b). For second-order differential operators this procedure has now become automated [28]. For fourth-order differential operators considerably less is known [29].

In practice, we will see that only the first Seeley-DeWitt coefficients are germane to the problem and that it is sufficient to find the "integrated" Seeley-DeWitt coefficients. This permits us to calculate the relevant coefficients by means of a technique based on the Feynman-Hellman formula [30], which can itself be viewed as a specialization of the Baker-Campbell-Hausdorff formula [31].

We regulate the one-loop effective action by cutting off the lower limit of the proper time integral

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi] \equiv -\frac{\mathcal{A}}{2} \int_\epsilon^{+\infty} \frac{\mathrm{d}s}{s} \text{Tr}(e^{-sH_{\text{field}}} - e^{-s[H_0]_{\text{field}}})$$

$$+ \frac{\mathcal{A}}{2} \int_\epsilon^{+\infty} \frac{\mathrm{d}s}{s} \text{Tr}(e^{-sH_{\text{ghost}}} - e^{-s[H_0]_{\text{ghost}}}), \qquad (37)$$

where we can identify $\epsilon = 1/\Omega_{\text{cut-off}}^2$ and $\Gamma_\epsilon^{(1-\text{loop})}[\phi]$ with $\Gamma_{\Omega_{\text{cut-off}}}^{(1-\text{loop})}[\phi]$. As the product sH must be dimensionless, we deduce that s has engineering dimensions of $(\text{time})^2$ or equivalently, $(\text{frequency})^{-2}$. In this stochastic field theory the cut-off can be taken to be a frequency scale $\Omega_{\text{cut-off}}$, and this identification allows us to compare between these two types of cut-off (proper time versus frequency). Since these theories are *not* Lorentz invariant, a frequency cut-off is not "quite" interchangeable with a wavenumber cut-off (more on this point below).

Substituting the above ansatz equations (35a)–(35b) into Eq. (37), making use of the explicit form for G_{free} Eq. (33), and expanding out the first terms of f_{field} and f_{ghost} yields

$$\frac{\Gamma_\epsilon^{(1-\text{loop})}[\phi]}{\mathcal{A}} = -\frac{1}{2} \int_\epsilon^{+\infty} \frac{\mathrm{d}s}{s} \int \mathrm{d}^n x \left[G_{\text{free}}(x,x'|s)\right]$$

$$\times ([f_{\text{field}}(x,x'|s)] - [f_{\text{ghost}}(x,x'|s)])$$

$$= -\frac{1}{2} \int_\epsilon^{+\infty} \frac{\mathrm{d}s}{s} \int \mathrm{d}^n x \left[G_{\text{free}}(x,x'|s)\right]$$

$$\times (s^{1/2}[b_{1/2}(x,x')] - s^{1/2}[a_{1/2}(x,x')]$$

$$+ s[b_1(x,x')] - s[a_1(x,x')] + O(s^{3/2}))$$

$$= -\frac{1}{2} A_d \int_\epsilon^{+\infty} \frac{\mathrm{d}s}{s^{3/2+d/4}} \int \mathrm{d}^n x\, (s^{1/2}[c_{1/2}(x,x')]$$

$$+ s[c_1(x,x')] + O(s^{3/2})). \qquad (38)$$

In arriving at this last expression we have used the fact that the coincidence limit of the free heat kernel is

$$[G_{\text{free}}(x,x'|s)] = A_d s^{-1/2-d/4},\tag{39}$$

which follows immediately from Eq. (33). We have also made use of the standard notation to express coincidence limits. Given any function $h(x,x')$, we write

$$\lim_{x'\to x} h(x,x') = [h(x,x')].\tag{40}$$

(Although we also employ the square brackets to denote arguments of functionals and functions, and to group expressions, the intended meaning should be clear from context and there should be no confusion.) We have defined *net* Seeley-DeWitt coefficients

$$c_{l/2} \equiv b_{l/2} - a_{l/2}, \quad \forall\ l = 0,1,2,\dots\ .\tag{41}$$

The Seeley-DeWitt coefficients $b_{l/2}$, $a_{l/2}$, and $c_{l/2}$ are functions of the mean field $\phi(\vec{x},t)$ and its derivatives, and as remarked above, can, in principle, be determined by solving a recursion relation resulting from inserting the ansatz equations (35a) and (35b) into Eq. (30). However, to obtain the *form* of the divergences of the one-loop effective action we need not evaluate these coefficients. It suffices to calculate the lower bound $(s\to0)$ of the proper time integral. In the limit $\epsilon\to0$ we find that the divergent terms in the RD effective action are given by

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi] = -\frac{1}{2}A_d\mathcal{A}\left(\frac{4}{d}\,\epsilon^{-d/4}\int\ d^n x\,[c_{1/2}(x,x')]\right.$$

$$-\frac{\epsilon^{1/2-d/4}}{\left(\frac{1}{2}-\frac{d}{4}\right)}\int\ d^n x\,[c_1(x,x')]$$

$$\left. -\frac{\epsilon^{1-d/4}}{\left(1-\frac{d}{4}\right)}\int\ d^n x\,[c_{3/2}(x,x')] + \cdots\right).$$

$$\tag{42}$$

We now list the divergences in the RD one-loop effective action for the following space dimensions

$$\underline{d=0}\quad \Gamma_\epsilon^{(1-\text{loop})}[\phi] = -2A_0\,\mathcal{A}\log(\Omega^2\,\epsilon)\int\ dt\,[c_{1/2}],\tag{43a}$$

$$\underline{d=1}\quad \Gamma_\epsilon^{(1-\text{loop})}[\phi] = -2A_1\,\mathcal{A}\,\epsilon^{-1/4}\int\ dx\,dt\,[c_{1/2}],\tag{43b}$$

$$\underline{d=2}\quad \Gamma_\epsilon^{(1-\text{loop})}[\phi] = -\frac{A_2}{2}\mathcal{A}\left(2\epsilon^{-1/2}\int\ d^2\vec{x}\,dt\,[c_{1/2}]\right.$$

$$\left. -\log(\Omega^2\,\epsilon)\int\ d^2\vec{x}\,dt\,[c_1]\right),$$

$$\tag{43c}$$

$$\underline{d=3}\quad \Gamma_\epsilon^{(1-\text{loop})}[\phi] = -\frac{A_3}{2}\mathcal{A}\left(\frac{4}{3}\,\epsilon^{-3/4}\int\ d^3\vec{x}\,dt\,[c_{1/2}]\right.$$

$$\left. +4\epsilon^{-1/4}\int\ d^3\vec{x}\,dt\,[c_1]\right).\tag{43d}$$

In all of these cases we only need to solve for the first two adiabatic (Seeley-DeWitt) coefficients $c_{1/2}$, and c_1; indeed, it is only the spacetime integrated *net* coefficients that are needed. In higher *space* dimensions, *additional* $c_{1/2}$'s would be required. However, for most practical applications it is enough to consider $0\le d\le3$. (Moreover, earlier work regarding the one-loop effective potential for RD indicates that this field theory is non-renormalizable for $d\ge4$ [5].) This dimension range covers the spatially homogeneous limit ($d=0$), one-dimensional (linear) systems ($d=1$), surfaces ($d=2$), and bulk systems (volumes) ($d=3$). We see that the divergences are of two types: (fractional) powers of the cutoff and logarithms of the cut-off. Only the latter can yield one-loop renormalization group beta functions and associated RGEs.

IV. CALCULATION OF THE SEELEY-DEWITT COEFFICIENTS

In this section we present a formalism that can, *in principle*, yield all the Seeley-DeWitt coefficients. As we have seen in the previous section, the calculation of the one-loop effective action involves solving auxiliary partial differential equations of parabolic type (denoted as heat equations, even if the diffusion is nonstandard).

In this formalism (see Appendix C) the first step is to write

$$\text{Tr}[\exp(-sH)] = \text{Tr}\{\exp[-s(H_{\text{free}}+\delta H)]\},\tag{44}$$

where δH is a lower-order differential operator when compared to H or H_{free}. We now apply a version of the Feynman-Hellman formula [30,31], as discussed in Appendix C, to obtain

$$\text{Tr}[\exp(A+\epsilon B)] = \text{Tr}[\exp(A)] + \epsilon\,\text{Tr}[B\exp(A)]$$

$$+\frac{\epsilon^2}{2}\int_0^1 d\ell\,\text{Tr}\{B\exp(\ell A)\,B$$

$$\times\exp[(1-\ell)A]\} + O(\epsilon^3).\tag{45}$$

This equation will be the basis for extracting the first two *integrated* Seeley-DeWitt coefficients.

The second step is to realize that we only need to look at the difference

$$\text{Tr}[\exp(-sH_{\text{field}})] - \text{Tr}[\exp(-sH_{\text{ghost}})], \qquad (46)$$

and write

$$\text{Tr}[\exp(-sH_{\text{ghost}})] = \text{Tr}\{\exp[-s(H_{\text{free}} + \delta H_1)]\}, \qquad (47a)$$

$$\text{Tr}[\exp(-sH_{\text{field}})] = \text{Tr}\{\exp[-s(H_{\text{free}} + \delta H_1 + \delta H_2)]\}. \qquad (47b)$$

If we take the difference of the previous operators, the $O(\epsilon^0)$ term automatically cancels, as does the $O[\epsilon^2(\delta H_1)^2]$ term, leaving

$$\text{Tr}[\exp(-sH_{\text{field}})] - \text{Tr}[\exp(-sH_{\text{ghost}})]$$
$$= -\epsilon s \text{Tr}[\delta H_2 \exp(-sH_{\text{free}})]$$
$$+ \frac{\epsilon^2}{2} s^2 \int_0^1 d\ell \, \text{Tr}\{\delta H_2 \exp(-\ell s H_{\text{free}})$$
$$\times \delta H_2 \exp[(1-\ell)sH_{\text{free}}])\}$$
$$+ \epsilon^2 s^2 \int_0^1 d\ell \, \text{Tr}\{\delta H_1 \exp(-\ell s H_{\text{free}})$$
$$\times \delta H_2 \exp[(1-\ell)sH_{\text{free}}])\} + O(\epsilon^3). \qquad (48)$$

We now make use of the explicit form of these "Hamiltonians." We write

$$H_{\text{ghost}} = (D^\dagger - V')(D - V')$$
$$= D^\dagger D - (D^\dagger - V')V' + 2\nu\nabla V' \cdot \nabla, \qquad (49)$$

where in the last term of the right hand side both ∇'s act on *everything* to the right. We know that the free Hamiltonian is given by

$$H_{\text{free}} = D^\dagger D = [-\partial_t^2 + \nu^2(\nabla^2)^2], \qquad (50)$$

so that we can identify δH_1 as

$$\delta H_1 = -[(D^\dagger - V')V' - 2\nu\nabla V' \cdot \nabla]. \qquad (51)$$

(Note that δH_1 is a linear differential operator.) From the definition of the ghost Hamiltonian

$$H_{\text{field}} = H_{\text{ghost}} - V''(D\phi - V), \qquad (52)$$

we deduce the following form for δH_2

$$\delta H_2 = -V''(D\phi - V). \qquad (53)$$

(Note that δH_2 is a function, not a differential operator.) Now consider the first order perturbation [the $O(\epsilon)$ term]

$$X_1 \equiv s \, \text{Tr}\{[V''(D\phi - V)] \exp(-sH_{\text{free}})\}. \qquad (54)$$

From the known form of the free kernel [see, e.g., Eqs. (33) or (A11)], and the fact that δH_2 is a function, this reduces to

$$X_1 = A_d s^{-1/2-d/4} \int d^n x \, [s \, V''(D\phi - V)]. \qquad (55)$$

This implies that the first-order perturbation does not contribute to the Seeley-DeWitt coefficient $c_{1/2}$, though it does contribute to c_1. In fact, we can write

$$\int d^n x \, [c_1] = \int d^n x \, V''(D\phi - V) + \cdots. \qquad (56)$$

This is actually the *only* contribution to the relevant Seeley-DeWitt coefficients. (There might have been additional contributions coming from those portions of the second-order term X_2 that have space derivatives; fortunately they vanish, as we now verify.) Let us consider

$$X_2 \equiv \frac{s^2}{2} \int_0^1 d\ell \, \text{Tr}\{[V''(D\phi - V)]$$
$$\times \exp(-s\ell H_{\text{free}})[V''(D\phi - V)]$$
$$\times \exp[-s(1-\ell)H_{\text{free}}]\}$$
$$+ s^2 \int_0^1 d\ell \, \text{Tr}\{[V''(D\phi - V)]$$
$$\times \exp(-s\ell H_{\text{free}})[(D^\dagger - V')V'$$
$$- 2\nu(V'\nabla^2 + (\nabla V') \cdot \nabla)] \exp[-s(1-\ell)H_{\text{free}}]\}. \qquad (57)$$

We can have any of the following cases:

• No gradients hit the free kernel: the term containing two factors of $[V''(D\phi - V)]$ is proportional to s^2 and can only contribute to c_2, which is not needed in the present context.

• One gradient hits one kernel: from Eqs. (33) or (A11), one can see that there is a factor of $(x-x')_i/(\nu\sqrt{s})$ of order $s^{3/2}$. Such a term is odd under the interchange of x and x' and will vanish when taking the spacetime trace.

• Two gradients hit the same free kernel: there will be contributions of the type

$$C_{d,1} \frac{\delta_{ij}}{\nu\sqrt{s}} + (C_{d,1})^2 \frac{(x-x')_i (x-x')_j}{d^2 s}, \qquad (58)$$

which, after tracing with the free kernel, yield contributions proportional to $s^{3/2}$. Therefore, these terms contribute to $c_{3/2}$, which is not needed.

Continuing in this manner, it is easy to convince oneself that there are no additional contributions to the required coefficients $c_{1/2}$ and c_1. We can finally write

$$\int d^n x \, [c_{1/2}] = 0, \qquad (59a)$$

$$\int d^n x \, [c_1] = \int d^n x \, [V''(D\phi - V)]. \qquad (59b)$$

Note that the present calculation only yields the *integrated on-diagonal* $(x=x')$ Seeley-DeWitt coefficients and is insensitive to any term that vanishes upon integration.

With a little more work along these lines, it is also possible to obtain the Seeley-DeWitt coefficients: $[a_{1/2}]$ and

$[a_1]$ for the ghost field, and $[b_{1/2}]$ and $[b_1]$ for the physical field. We only quote the results here

$$\int d^n x [a_{1/2}] = \int d^n x \, 2 C_{d,1} V', \qquad (60a)$$

$$\int d^n x [a_1] = \int d^n x \left[\frac{d}{2} (V')^2 + (D^\dagger - V') V' \right], \quad (60b)$$

$$\int d^n x [b_{1/2}] = \int d^n x \, 2 C_{d,1} V', \qquad (60c)$$

$$\int d^n x [b_1] = \int d^n x \left[\frac{d}{2} (V')^2 + (D^\dagger - V') V' + V''(D\phi - V) \right]. \qquad (60d)$$

V. ONE-LOOP RENORMALIZATION

We have already calculated the (regularized) one-loop effective action equation (20) for the RD equation, and thus, we may now explore the renormalizability of this field theory following the prescription reviewed in Ref. [20]. In order to do so we must analyze the divergences of the one-loop effective action $\Gamma_\epsilon^{(1-\text{loop})}$. We must also keep in mind the fact that the bare theory [i.e., defined by the action equation (12)] does not depend on the arbitrary scale μ introduced by the renormalization scheme. Therefore, just as for the case of QFTs [18,19], we will derive a set of equations that govern the scale dependence of the parameters appearing in the RD effective action from the identity

$$\mu \frac{d\Gamma[\phi]}{d\mu} = 0$$

$$= \mu \frac{d(S[\phi] - S[v_0])}{d\mu} + \mu \frac{d\Gamma_\epsilon^{(1-\text{loop})}[\phi]}{d\mu} + O(\mathcal{A}^2),$$

$$(61)$$

where the $O(\mathcal{A}^2)$ indicates there will be higher-loop contributions to the effective action. In quantum field theory this identity does yield the one-loop RGEs since Eq. (61) can be expressed in terms of a sum of independent field operators (operator basis) and each coefficient of an element of this basis determines an independent RGE.

As we have already calculated the relevant Seeley-DeWitt coefficients for the RD equation, we now turn to investigate the one-loop renormalizability of its stochastic field theory. The renormalizability criteria are based on the following definitions. For renormalizable and super-renormalizable theories the counterterms needed to cancel the divergences are equal to, or fewer in number than the terms appearing in the zero-loop action, which implies that the Seeley-DeWitt coefficients are expandable in terms of the *same* operator basis appearing in the classical action. In particular, this basis set consists of $\{\partial_t \phi, \nabla^2 \phi, 1, \phi, \phi^2, \ldots, \phi^N\}$. For nonrenormalizable theories this criterion fails. That is, there are terms in the integrated Seeley-DeWitt coefficients that do not appear in the classical action [14,15,32,33].

By comparing the zero-loop action equation (12) with the divergent terms of the one-loop effective action, it is easy to see that the latter do not contain any field operators not already present in the bare (classical) action. The divergent contributions to the one-loop effective action in d-dimensions are given by

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi; v_0] = \frac{\mathcal{A}}{2} A_d \frac{\epsilon^{1/2 - d/4}}{\left(\frac{1}{2} - \frac{d}{4} \right)} \int d^d x \, dt [c_1(\phi) - c_1(v_0)]$$

$$+ O(\epsilon^{1 - d/4})$$

$$= \frac{\mathcal{A}}{2} A_d \frac{\epsilon^{1/2 - d/4}}{\left(\frac{1}{2} - \frac{d}{4} \right)} \int d^d x \, dt \{ V''(\phi)$$

$$\times [D\phi - V(\phi)] + V''(v_0) V(v_0) \}$$

$$+ O(\epsilon^{1 - d/4}). \qquad (62)$$

Some remarks are in order. First of all we point out the fact that the one-half Seeley-DeWitt coefficients of the field and ghost mutually cancel out. This cancellation is special to the RD system and does not take place for generic SPDEs. Secondly, the ill-defined quantity ''$\epsilon^0/0$'' arising in $d = 2$ must be replaced by $\log(\Omega^2 \epsilon) = \log(\Omega^2/\Omega_{\text{cut-off}}^2)$. The dimensionfull (but arbitrary) parameter Ω is required to make the argument of the logarithm dimensionless. It is often more convenient to introduce a cut-off in wavenumber, rather than in frequency. In Lorentz invariant theories (QFTs, for example) these are essentially equivalent and it is usual to adopt units where the speed of light is one. In the RD system this would be inappropriate, since the equation is not Lorentz invariant. Instead, one notes that dimensionally

$$[\epsilon] = [\text{proper time } s]$$

$$= [\text{physical time } t]^2$$

$$= [\nu]^{-2} [\text{distance}]^4, \qquad (63)$$

and therefore, in terms of a wavenumber cut-off Λ and a wavenumber subtraction point μ

$$``\epsilon^0/0'' \rightarrow \log(\Omega^2 \epsilon) = \log(\Omega^2/\Omega_{\text{cut-off}}^2) = \log(\mu^4/\Lambda^4). \qquad (64)$$

(This observation is important when comparing different regularization schemes; for instance the effective potential calculation of Ref. [5] uses a wavenumber cut-off.)

Thirdly, it is of great importance to study the *type* of divergence arising in the one-loop effective action for the RD equation, i.e., logarithms versus (fractional) power. From the above we see that the type of divergence depends on the number of *space* dimensions d. If d is odd, there will never be logarithmic divergences to one-loop order in the RD field theory; to get a logarithm, the space dimensionality must be even. A similar feature holds also for the one-loop effective action for QFTs in odd *spacetime* dimensions [20,33]. We can conclude that the appearance of logarithmic divergences

for specific space dimensions is not an artifact of the RD field theory, nor of SPDEs, nor is it an artifact of the regularization scheme we have employed. In QFT the RGEs yield the running of the coupling constants, i.e., give the scale dependence, if and only if, there are logarithmic divergences in the effective action. Thus, we can already predict that the parameters in the RD equation will not run in the ultraviolet region (to one-loop order) for *odd* space dimensions [33].

Nevertheless, the one-loop effective action in odd space dimensions still contains divergences (though not for $d=1$), which must be subtracted by suitable counterterms, but once this subtraction has been performed, the remaining finite part $\Gamma_{\text{finite}}^{(1-\text{loop})}$ is independent of the subtraction scale μ.

VI. RENORMALIZATION OF THE RD EFFECTIVE ACTION

In this section we calculate the counterterms needed to renormalize the one-loop effective action. We first start with the bare classical action equation (12) for the reaction-diffusion equation

$$S[\phi] = \tfrac{1}{2} \int d^n x (\partial_t \phi - \nu \nabla^2 \phi - V[\phi])^2$$

$$= \tfrac{1}{2} \int d^n x \left(\partial_t \phi - \nu \nabla^2 \phi - \sum_{j=0}^{N} \frac{\lambda_j}{j!} \phi^j \right)^2. \quad (65)$$

The bare parameters (no subscript) are related to the renormalized ones (denoted by a subscript R) as follows

$$\phi = Z^{-1/2} \phi_R, \quad \text{with} \quad Z = 1 + \delta Z, \quad (66)$$

$$\lambda_j = \lambda_j^R + \delta \lambda_j, \quad (67)$$

$$\nu = \nu_R + \delta \nu, \quad (68)$$

with δZ, $\delta \lambda_j$, and $\delta \nu$ the corresponding counterterms for Z, λ_j, and ν, respectively. (Our convention for the definition of the wavefunction renormalization constant Z does not follow the standard one in QFT [14,15].)

Our task consists in demonstrating that all the divergences appearing in the regulated one-loop effective action can be cancelled by suitable choices for these counterterms. In effect, we absorb the divergences into the (bare) parameters of the RD equation by *renormalizing* these parameters. If we write the bare action in terms of the renormalized parameters and counterterms, and keep up to linear order in the counterterms (which is sufficient for a one-loop analysis), we find

$$S[\phi] = \tfrac{1}{2} \int d^n x \left(\partial_t \phi_R - \nu_R \nabla^2 \phi_R - \sum_{j=0}^{N} \frac{\lambda_j^R}{j!} \phi_R^j \right)^2$$

$$- \frac{\delta Z}{2} \int d^n x \left(\partial_t \phi_R - \nu_R \nabla^2 \phi_R - \sum_{j=0}^{N} \frac{\lambda_j^R}{j!} \phi_R^j \right)^2$$

$$- \int d^n x \left(\partial_t \phi_R - \nu_R \nabla^2 \phi_R - \sum_{j=0}^{N} \frac{\lambda_j^R}{j!} \phi_R^j \right)$$

$$\times \left(\delta \nu \nabla^2 \phi_R + \sum_{j=0}^{N} \frac{\phi_R^j}{j!} \left[\delta \lambda_j + \delta Z \frac{\lambda_j^R(1-j)}{2} \right] \right),$$

which can be written in a more compact and transparent notation as follows

$$S[\phi] = \tfrac{1}{2} \int d^n x (D_R \phi_R - V_R[\phi_R])^2$$

$$- \frac{\delta Z}{2} \int d^n x (D_R \phi_R - V_R[\phi_R])^2$$

$$- \int d^n x (D_R \phi_R - V_R[\phi_R]) \left(\delta \nu \nabla^2 \phi_R + \sum_{j=0}^{N} \frac{\phi_R^j}{j!} \right.$$

$$\left. \times \left[\delta \lambda_j + \delta Z \frac{\lambda_j^R(1-j)}{2} \right] \right) \quad (69)$$

$$= S_R[\phi_R, \mu] + S_\delta[\phi_R], \quad (70)$$

where we have introduced the scale dependent renormalized action $S_R[\phi_R, \mu]$

$$S_R[\phi_R, \mu] \equiv \tfrac{1}{2} \int d^n x (Z^{1/2}(\mu) \partial_t \phi - \nu_R(\mu) Z^{1/2}(\mu) \nabla^2 \phi$$

$$- V[Z^{1/2}\phi, \lambda_j^R(\mu)])^2, \quad (71)$$

with

$$V_R[\phi_R] \equiv V[Z^{1/2}\phi, \lambda_j^R(\mu)] = \sum_{j=0}^{N} \frac{\lambda_j^R(\mu)}{j!} Z^{j/2}(\mu) \phi^j,$$

$$(72)$$

the renormalized (scale dependent) potential.

The meaning of D_R is clear from inspection. The final equality in Eq. (70) defines the finite, renormalized action S_R and the divergent (but regulated) counterterm action S_δ. The individual counterterms appearing in S_δ will be used to cancel off the divergences arising in Eq. (62). We carry out this cancellation separately in each space dimension since each case leads to structurally different divergences [see Eq. (62)].

A. $d=0$ counterterms and renormalization

The case $d=0$ is very simple: in zero space dimensions there is no diffusion. There is a brief discussion in Ref. [5] and we do not belabor the point here except to reiterate that in $d=0$ the RD equation is one-loop renormalizable and one-loop finite.

B. $d = 1$ counterterms and renormalization

In one space dimension the (formally) divergent effective action is given by

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi_R] = -2\mathcal{A}A_1\epsilon^{-1/4}\int dx\, dt [c_{1/2}] + O(\mathcal{A}^2), \tag{73}$$

with $A_1 = \Gamma(1/4)/[8\pi(\pi\nu_R)^{1/2}]$. We did not explicitly write this term in Eq. (62) since it vanishes identically.

From our previous calculation of the Seeley-DeWitt coefficients we know that $[c_{1/2}] = 0$ (in all dimensions), which tells us that in one space dimension there are no divergences, that is, the theory is one-loop finite and there is no need to introduce counterterms. Since no renormalization is required, there will be no scale dependence in the parameters appearing in the RD equation. The beta functions, $\beta_\mathcal{O} \equiv \mu\, d\mathcal{O}/d\mu$ (encoding the scale dependence of the parameters) are therefore zero [at least up to order $O(\mathcal{A}^2)$], and we have

$$Z = 1 + O(\mathcal{A}^2), \tag{74}$$

$$\nu = \nu_R + O(\mathcal{A}^2), \tag{75}$$

$$\lambda_j = \lambda_j^R + O(\mathcal{A}^2). \tag{76}$$

C. $d = 2$ counterterms and renormalization

In two space dimensions the divergent effective action is given by Eq. (62)

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi_R] = +\frac{\mathcal{A}A_2}{2}\log(\mu^4/\Lambda^4)\int d^2\vec{x}\, dt [c_1], \tag{77}$$

where $A_2 = 1/(16\pi\nu_R)$.

From the calculation of the Seeley-DeWitt coefficients we know that for $d = 2$

$$[c_{1/2}] = 0, \tag{78a}$$

$$[c_1] = V_R''[\phi_R](D\phi_R - V_R[\phi_R]), \tag{78b}$$

where we have written the Seeley-DeWitt coefficient $[c_1]$ in terms of the renormalized parameters as we are only working to one-loop order. Therefore, for the divergent part of effective action we can write

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi_R] = \frac{\mathcal{A}}{8\pi\nu_R}\log(\mu/\Lambda)\int d^2\vec{x}\, dt\, V_R''[\phi_R]$$
$$\times (D_R\phi_R - V_R[\phi_R]). \tag{79}$$

In order to determine the value of the counterterms and to cancel them off, we must set

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi_R] = -S_\delta[\phi_R] + \text{finite}. \tag{80}$$

This cancellation can be made up to a residual finite but scale dependent logarithm. This is because the difference of two divergent logarithms can be finite and *nonzero*. The counterterms are proportional to $\log(\mu/\Lambda)$, where μ is an arbitrary

scale needed to render the argument dimensionless, but this scale need not coincide with μ_0, the *other* arbitrary scale needed to render the argument of the other logarithm, $\log(\mu_0/\Lambda)$, dimensionless [33].

If we perform this cancellation, we obtain the following μ-dependent family of solutions for the counterterms $\delta\lambda_j$

$$\frac{\mathcal{A}}{2}A_2\log(\mu^2/\Lambda^2)V_R''[\phi]$$

$$= \frac{\mathcal{A}}{2}A_2\log(\mu^2/\Lambda^2)\sum_{j=0}^{N}\frac{\lambda_j^R}{j!}j(j-1)\phi^{j-2}$$

$$= \sum_{j=0}^{N}\frac{\delta\lambda_j}{j!}\phi^j. \tag{81}$$

As we are working to one-loop order, we can set $Z(\mu)$ equal to one in V_R''. We can then read off the individual counterterms from this equation, using the linear independence of the basis $\{\partial_t\phi, \nabla^2\phi, 1, \phi, \phi^2, \ldots, \phi^N\}$, to obtain

$$\delta Z = 0 + O(\mathcal{A}^2), \tag{82a}$$

$$\delta\nu = 0 + O(\mathcal{A}^2), \tag{82b}$$

$$\delta\lambda_0 = \frac{\mathcal{A}}{8\pi\nu_R}\log(\mu/\Lambda)\lambda_2^R + O(\mathcal{A}^2), \tag{82c}$$

$$\delta\lambda_1 = \frac{\mathcal{A}}{8\pi\nu_R}\log(\mu/\Lambda)\lambda_3^R + O(\mathcal{A}^2), \tag{82d}$$

$$\vdots$$

$$\delta\lambda_{N-2} = \frac{\mathcal{A}}{8\pi\nu_R}\log(\mu/\Lambda)\lambda_N^R + O(\mathcal{A}^2), \tag{82e}$$

$$\delta\lambda_{N-1} = 0 + O(\mathcal{A}^2), \tag{82f}$$

$$\delta\lambda_N = 0 + O(\mathcal{A}^2). \tag{82g}$$

In particular, we see that there is no wavefunction renormalization nor diffusion constant renormalization in two dimensions at one-loop order. The couplings associated with the highest and next-to-highest powers of the field ($\lambda_{N-1}^R, \lambda_N^R$) do not require renormalization to this order.

As pointed out above, due to the logarithmic divergence in two dimensions, when we subtract the divergences from the counterterm action, we are left with a finite μ-dependent logarithmic piece in addition to the renormalized action, that is

$$\Gamma[\phi] = S[\phi] + \Gamma_\epsilon^{(1-\text{loop})}[\phi] + \Gamma_{\text{finite}}^{(1-\text{loop})}[\phi]$$

$$= S_R[\phi_R, \mu] + S_\delta[\phi_R] + \Gamma_\epsilon^{(1-\text{loop})}[\phi_R]$$

$$+ \Gamma_{\text{finite}}^{(1-\text{loop})}[\phi_R]$$

$$= S_R[\phi_R, \mu] + \frac{\mathcal{A}A_2}{2}\log\left(\frac{\mu_0^4}{\mu^4}\right)\int d^2\vec{x}\, dt (D_R\phi_R$$

$$- V_R[\phi_R])V_R''[\phi_R] + O(\mathcal{A}^2). \tag{83}$$

We insert this expression into Eq. (61) to obtain the one-loop RGE

$$\mu \frac{d}{d\mu}(D_R \phi_R - V_R[\phi_R]) = \frac{\mathcal{A}}{8\pi\nu_R} V_R''[\phi_R] + O(\mathcal{A}^2). \tag{84}$$

In arriving at this equation, we have cancelled an overall *common* factor of the classical equation of motion, since $D_R \phi_R - V_R[\phi_R] \neq 0$ in general.

By collecting up the coefficients of the linearly independent terms in Eq. (84) we find the corresponding one-loop RGEs and beta functions in $d=2$ to be given by

$$\beta_Z = \mu \frac{dZ}{d\mu} = 0 + O(\mathcal{A}^2), \tag{85a}$$

$$\beta_\nu = \mu \frac{d\nu_R}{d\mu} = 0 + O(\mathcal{A}^2), \tag{85b}$$

$$\beta_{\lambda_0} = \mu \frac{d\lambda_0^R}{d\mu} = -\frac{\mathcal{A}}{8\pi\nu_R} \lambda_2^R + O(\mathcal{A}^2), \tag{85c}$$

$$\beta_{\lambda_1} = \mu \frac{d\lambda_1^R}{d\mu} = -\frac{\mathcal{A}}{8\pi\nu_R} \lambda_3^R + O(\mathcal{A}^2), \tag{85d}$$

$$\vdots$$

$$\beta_{\lambda_{N-2}} = \mu \frac{d\lambda_{N-2}^R}{d\mu} = -\frac{\mathcal{A}}{8\pi\nu_R} \lambda_N^R + O(\mathcal{A}^2), \tag{85e}$$

$$\beta_{\lambda_{N-1}} = \mu \frac{d\lambda_{N-1}^R}{d\mu} = 0 + O(\mathcal{A}^2), \tag{85f}$$

$$\beta_{\lambda_N} = \mu \frac{d\lambda_N^R}{d\mu} = 0 + O(\mathcal{A}^2). \tag{85g}$$

Since there is no wavefunction nor diffusion constant renormalization the set of one-loop RGEs can be summarized in the following way

$$\mu \frac{dV_R[\phi_R]}{d\mu} = -\frac{\mathcal{A}}{8\pi\nu_R} V_R''[\phi_R] + O(\mathcal{A}^2). \tag{86}$$

This equation agrees with the computation of the RGEs based on the effective potential, which was calculated in Ref. [5]. [See Eq. (51) of that paper.] Furthermore, if we define $\mu = \mu_0 \exp(\tau)$, the previous RGE becomes

$$\frac{dV_R[\phi_R]}{d\tau} = -\frac{\mathcal{A}}{8\pi\nu_R} \frac{d^2 V_R[\phi_R]}{d\phi_R^2} + O(\mathcal{A}^2), \tag{87}$$

which implies the fact that the one-loop RGE in $d=2$ behaves like an anti-diffusion process in field space.

At this point it is interesting to compare our results with independent renormalization group results obtained, for example, by Cardy in Ref. [6]. If a path integral is derived (along the lines given in Refs. [21–23]) for the two-body process $A+A \to$ inert, then the corresponding RD equation turns out to be given by [6]

$$D\phi = -2\lambda\phi^2 + \eta(x), \tag{88}$$

where however, the noise must be pure imaginary. A renormalization group analysis of Eq. (88) shows that the field ϕ does not require wavefunction renormalization, nor does the diffusion constant ν renormalize. Our one-loop heat kernel computation performed for an *arbitrary* reaction polynomial, Eqs. (85a) and (85b), is in complete accord with these results (even though we treat real noise). Returning to Eq. (88), the only nonvanishing renormalization is that of the coupling λ. It turns out that the one-loop renormalization group beta function is exact, and when expressed in terms of the dimensionless renormalized coupling g_R is given by Ref. [6]

$$\beta(g_R) = b g_R^2, \tag{89}$$

in $d=2$ dimensions, where b is a positive constant (the value of this constant is not specified in Ref. [6]).

In order to compare these results, we define the following dimensionless couplings

$$g_j \equiv \frac{\mathcal{A}}{8\pi\nu_R} \frac{\lambda_{j+2}^R}{\lambda_j^R}, \quad 0 \leq j \leq N-2, \tag{90}$$

provided, of course, that for a given j the coupling constant $\lambda_j^R \neq 0$. This definition together with the hierarchy of beta functions given in Eq. (85) show that the dimensionless coupling constants g_j satisfy the following one-loop RGE

$$\beta(g_j) \equiv \mu \frac{dg_j}{d\mu} = g_j \left(\frac{\dot{\lambda}_{j+2}^R}{\lambda_{j+2}^R} - \frac{\dot{\lambda}_j^R}{\lambda_j^R} \right), \tag{91}$$

where the overdot is shorthand notation for $\mu d/d\mu$. We now consider an RD equation of the type given in Eq. (1) with real noise and for a degree-two ($N=2$) reaction polynomial equation (3) $V[\phi] = \lambda_0 + \lambda_2/2\phi^2$. This particular choice is made in order to be able to treat an RD equation as close as possible in structure to the one in Eq. (88). Apart from the imaginary versus real noise, the essential difference lies in the fact that we (must) have a tadpole term, whereas Eq. (88) lacks such a term. In this case there is only one dimensionless coupling which can be defined, namely

$$g_0 = \frac{\mathcal{A}}{8\pi\nu_R} \frac{\lambda_2^R}{\lambda_0^R}, \tag{92}$$

and Eq. (91) implies the following one-loop beta function for this dimensionless coupling

$$\beta(g_0) = g_0 \left(\frac{\dot{\lambda}_2^R}{\lambda_2^R} - \frac{\dot{\lambda}_0^R}{\lambda_0^R} \right) = -g_0 \left(\frac{\dot{\lambda}_0^R}{\lambda_0^R} \right) = g_0^2. \tag{93}$$

This follows from Eq. (85c) and from the fact that $\dot{\lambda}_2^R \propto \lambda_4^R = 0$.

Thus, for the purposes of renormalization and calculating the RGEs, this example demonstrates that it is equivalent to start from a complex or real SPDE, and that the field theory can be derived from microscopic principles or obtained by means of the procedure outlined in Ref. [4].

D. $d=3$ counterterms and renormalization

In three space dimensions the divergent effective action is given by Eq. (62)

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi_R] = -2\mathcal{A}A_3\epsilon^{-1/4}\int d^3\vec{x}\,dt[c_1], \qquad (94)$$

with $A_3 = \Gamma(3/4)/[16\pi(\pi\nu_R)^{3/2}]$.

The net Seeley-DeWitt coefficient is given by

$$[c_1] = V''[\phi_R](D\phi_R - V_R[\phi_R]). \qquad (95)$$

The vanishing of the index one-half Seeley-DeWitt coefficient means that the divergent effective action in $d=3$ becomes

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi_R] = -2\mathcal{A}A_3\epsilon^{-1/4}\int d^3\vec{x}\,dt\,V''[\phi_R]$$
$$\times(D_R\phi_R - V[\phi_R]) + O(\mathcal{A}^2). \qquad (96)$$

In order to determine the value of the counterterms we must once again set

$$\Gamma_\epsilon^{(1-\text{loop})}[\phi_R] = -S_\delta[\phi_R] + \text{finite}. \qquad (97)$$

The last identification yields the following (μ-independent) set of counterterms

$$\delta Z = 0 + O(\mathcal{A}^2), \qquad (98\text{a})$$

$$\delta\nu = 0 + O(\mathcal{A}^2), \qquad (98\text{b})$$

$$\delta\lambda_0 = -\mathcal{A}A_3\lambda_2^R\epsilon^{-1/4} + O(\mathcal{A}^2), \qquad (98\text{c})$$

$$\delta\lambda_1 = -\mathcal{A}A_3\lambda_3^R\epsilon^{-1/4} + O(\mathcal{A}^2), \qquad (98\text{d})$$

$$\vdots$$

$$\delta\lambda_{N-2} = -\mathcal{A}A_3\lambda_N^R\epsilon^{-1/4} + O(\mathcal{A}^2), \qquad (98\text{e})$$

$$\delta\lambda_{N-1} = 0 + O(\mathcal{A}^2), \qquad (98\text{f})$$

$$\delta\lambda_N = 0 + O(\mathcal{A}^2). \qquad (98\text{g})$$

As there is no scale dependent logarithmic divergence at one-loop order in three-dimensions, all the beta functions vanish [33].

VII. DISCUSSION

In this paper we have generalized and applied a method based on the DeWitt-Schwinger proper time expansion to calculate the ultraviolet divergences of the one-loop effective action associated to the RD equation. This particular ap-

proach involves the physical degrees of freedom plus the ''ghost'' fields, which are needed to account for the functional Jacobian that arises from a certain change of variables [4]. For RDs this Jacobian is generally nonvanishing and must be taken into account in the computation of the characteristic functional. The importance of the effective action lies in the fact that it is the quantity needed to derive equations of motion, which correctly take into account fluctuations and interactions to a given number of loops. The effective action encodes the dynamics of the system. By contrast, the effective potential can only tell us about static solutions. In order to know whether the minima of the effective potential are relevant as solutions of the late time behavior of the system we must see how accessible these solutions are. But before any of these questions can be answered, the effective action must be calculated.

The one-loop effective action is given in terms of a functional determinant, which must be regulated and renormalized. The heat kernel technique is an established method (used in QFT) for carrying this out. In QFT the functional determinant appearing in the one-loop effective action is usually quadratic in both time (∂_t^2) and space derivatives (∇^2). In passing to a Euclidean spacetime, the corresponding proper time equation for the kernel to be solved Eq. (29) is a heat equation for diffusion in $n=d+1$ dimensions, with s playing the rôle of diffusion time. This is why its Green function (whether exactly or approximately calculated) is justifiably denoted as the heat kernel. However, for SPDEs, such as those considered in Ref. [4], the functional determinant in the corresponding one-loop effective action involves not only a ''mismatch'' between the number of spatial and temporal derivatives, but also fourth- or even higher-order spatial derivatives. We have seen an explicit example of this in the RD equation treated here, which yields second order in time but fourth order in space derivatives, $(\nabla^2)^2$, for its associated operator determinant. (Even higher derivatives will be encountered in the one-loop effective actions based on the Sivashinski and the Swift-Hohenberg equations: two time derivatives but eight spatial derivatives.) While much is known about the standard heat kernel and its associated Seeley-DeWitt coefficients, very few (as far as the authors are aware) of these ideas have been applied to other types of field theories whose fluctuations may be of a nonquantum nature (i.e., noise) [29]. The heat kernel technique (and its generalization) is well suited to regularize one-loop effective actions, and therefore, is very useful to handle theories with derivative-type interactions, as well as higher derivative ''kinetic'' terms.

We have applied these techniques to compute the one-loop effective action for the RD equation. As regards its ultraviolet renormalizability, we found that the terms appearing at one-loop order have the same structure, i.e., involve the same terms present already at the level of the ''classical'' or zero-loop action. Strictly speaking, this claim holds true only if a certain bare constant is added to the original equation of motion, as we have seen. This constant, or ''tadpole,'' is needed to carry out the one-loop renormalization of the leading divergence that appears in the effective action. Moreover, this constant admits a simple physical interpreta-

tion and can be ascribed either to a constant flux rate or as the mean value of the additive noise source. In regards to the *scale* of application of the RD equation, the one-loop renormalizability indicates that although RD is a macroscopic equation [only intended to make physical sense for scales greater than a certain minimum length scale L_0, defined by the underlying molecular physics (if one is discussing chemical reactions)], we have shown in this paper that the ultraviolet behavior of the RD equation is controlled, and that considered as a strictly phenomenological equation its short distance behavior is *much better than one had any right to expect*. The short distance structure of this effective action has been revealed via the calculation of the Seeley-DeWitt coefficients up to one-loop order in the noise amplitude. By means of this information, we have been able to establish the one-loop finiteness of RD equations driven by additive white noise for $d=0$ and $d=1$ space dimensions and their renormalizability for $d=2$ and $d=3$ space dimensions. In $d=2$ space dimensions there are logarithmic divergences which lead to running, (scale dependence) of the parameters that describe the RD equation. There is no wavefunction renormalization at one-loop order. These results hold for polynomial reaction kinetics of arbitrary degree N. (Note: The absence of wavefunction renormalization has already been demonstrated for the case of a (complex) RD equation describing pair reactions ($N=2$), where it turns out that the one-loop beta function in $d=2$ is exact [6].) When taken as a model for pattern development, this result becomes even more striking since this means we can safely use the RD equation to investigate the important short distance and small time limit of the field correlations that arise in pattern formation, as already remarked earlier.

The RGE results obtained here are identical to those derived (by different means) from an effective potential calculation for RD equations presented in Ref. [5]. The effective potential is the effective action restricted to constant field configurations and plays an important role in uncovering patterns of symmetry breaking and in the onset of pattern formation. Nevertheless, the claim of the one-loop renormalizability made in Ref. [5] requires the investigation of the wavefunction renormalization which was beyond the scope of that paper. The work presented here is also intended to complete and complement that discussion. Moreover, as pointed out there, the combined effects of noise and interactions is to shift the symmetric states of the system, as well as to change the nature of the linear instabilities that may be induced by perturbations around these new states. A spatial pattern is, by definition, a spatially inhomogeneous configuration with a higher or lesser degree of symmetry, if any such symmetry is at all present. Thus, to investigate the onset of the spatial-temporal patterns resulting from fluctuations and interactions, one must go beyond the effective potential. This requires working with inhomogeneous fields and the effective action.

The cautious reader will have noticed that most of the calculations developed here depend solely on the reaction potential V and its derivatives and not on the fact that V is a polynomial. In fact, this is easy to see from the structure of our Seeley-DeWitt coefficients Eqs. (59a)–(59b) and (60a)–

(60b). Thus, the question arises if our treatment can be extended and applied to handle RD equations with nonpolynomial reaction kinetics. The answer is in the affirmative provided that the potential admits a real solution to Eq. (18) (e.g., $V[\phi] \sim \sinh[\phi]$) since we need a constant background about which to expand the action, as indicated in Eq. (62). Provided such a solution exists, the rest of the analysis presented here carries through as is, up to the extraction of the renormalization group equations (which will again be nonvanishing only for $d=2$). At this point the explicit functional form of the potential changes the nature of the basis set of independent field operators that leads to the RGEs. Nonpolynomial reaction kinetics do indeed arise in many applications in chemical physics and in the modeling of biological pattern formation, typically in the form of rational functions (i.e., ratios of polynomials) and whenever *constraints* are to be imposed on the model [1,2]. When a coarse-grained field theoretic approach is applied to density waves in earthquakes, for example, a stochastic PDE for the (scalar) slip field (which measures the relative displacement of two elastic media in contact taken along the surface of contact) results which depends on the cosine of the slip field, $\cos \phi$, and is driven by additive white noise [34]. Thus, the work presented here is broad in scope.

Finally, these results also have the following practical application: as analytic calculations can be carried only so far, it is clear that numerical studies of SPDEs are crucial. Ultraviolet renormalizability corresponds to the situation in which long distance physics is largely insensitive to the details of short distance physics. In considering numerical studies of the RD equation, we can therefore assert the cut-off *insensitivity* of the numerical solutions, at least to one-loop. (In numerical studies, the ultraviolet cut-off is provided by the lattice spacing.) This is of paramount importance since a numerical study of RD will give us the information needed to see if the system thermalizes, if it has steady state solutions, and most importantly, if the minima of the effective potential calculated in Ref. [5] are explored in the time evolution of the system.

APPENDIX A: FREE GREEN FUNCTION FOR THE RD EFFECTIVE ACTION

In this Appendix we calculate the free Green function appearing in Eq. (33). The only ''difficult'' part of the analy-

sis is dealing with the fourth-order spatial bi-harmonic operator $(\nabla^2)^2$. Using translational invariance to set $x-x' \rightarrow x$, and introducing Fourier transforms $(d^n k = d^d k \, d\omega)$ as follows

$$G_{\text{free}}(\vec{x}t,\vec{0}0|s) = \int \frac{d^n k}{(2\pi)^n} \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \widetilde{G}_{\text{free}}(\vec{k}\omega|\Omega)$$

$$\times \exp[i(\vec{k}\cdot\vec{x}-\omega t-\Omega s)], \qquad (A1)$$

we easily find that

$$\widetilde{G}_{\text{free}}(\vec{k}\omega|\Omega) = \frac{1}{\omega^2+(\nu k^2)^2-i\Omega}, \qquad (A2)$$

where $k^2 = \vec{k}\cdot\vec{k}$. By inverting the Fourier transform equation (A1) one obtains the following integral representation for the free Green function

$$G_{\text{free}}(x,0|s) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} \int \frac{d\Omega}{2\pi} \frac{1}{\omega^2+(\nu k^2)^2-i\Omega}$$

$$\times \exp[i(\vec{k}\cdot\vec{x}-\omega t-\Omega s)]. \qquad (A3)$$

We first perform the integral over Ω by means of a contour integral in the complex Ω-plane. There is only one simple pole on the negative imaginary Ω-axis and we close the (semi-circular) contour (centered at the origin) in the lower half plane. As the radius of this arc goes to infinity, only the contribution along the real Ω-axis remains $(s>0)$. As a result and by the Residue Theorem we have (the contour is closed in the clockwise sense)

$$G_{\text{free}}(x,0|s) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} \exp[i(\vec{k}\cdot\vec{x}-\omega t)]$$

$$\times \exp(-s[\omega^2+(\nu k^2)^2]). \qquad (A4)$$

We can go further and compute the ω integral exactly to obtain

$$G_{\text{free}}(x,0|s) = \left(\frac{1}{4\pi s}\right)^{1/2} \exp\left(-\frac{t^2}{4s}\right) \int \frac{d^d k}{(2\pi)^d}$$

$$\times \exp[-s(\nu k^2)^2+i\vec{k}\cdot\vec{x}], \qquad (A5)$$

and the remaining momentum integral is manifestly convergent (for $d>0$). As for the boundary condition, note that for $s\rightarrow 0$, $\lim_{s\rightarrow 0} G_{\text{free}}(\vec{x}t,\vec{0}0|s) = \delta(t,0)\,\delta^d(\vec{x},\vec{0})$ (in the sense of distributions) as it must, since

$$\delta(t) \sim \lim_{s\rightarrow 0} \left(\frac{1}{4\pi s}\right)^{1/2} \exp[-t^2/(4s)], \qquad (A6)$$

and

$$\delta^d(\vec{x}) = \int \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}}. \qquad (A7)$$

One can also check that the boundary condition is satisfied, before integrating over ω, by simply setting $s=0$ in Eq. (A4). Let us now work out the momentum integration.

The angular integration is given by

$$\int d\Omega_{d-1} \exp(i\vec{k}\cdot\vec{x}) = (2\pi)^{d/2} \frac{J_{(d-2)/2}(kx)}{(kx)^{(d-2)/2}}. \qquad (A8)$$

The exact Green function or kernel for our free "heat" operator in Eq. (32) is

$$G_{\text{free}}(\vec{x}t,\vec{0}0|s) = \left(\frac{1}{4\pi s}\right)^{1/2} \exp\left(-\frac{t^2}{4s}\right)\frac{1}{(2\pi)^{d/2}}$$

$$\times \int_0^{+\infty} k^{d-1}dk \, \exp[-s(\nu k^2)^2]$$

$$\times \left[\frac{J_{(d-2)/2}(k|x|)}{(k|x|)^{(d-2)/2}}\right]. \qquad (A9)$$

This solves the differential equation (32) and satisfies the boundary condition $G_{\text{free}}(\vec{x},t|0) = \delta^d(\vec{x})\delta(t)$ for vanishing proper time s, hence this is the *unique* solution of Eq. (32).

Important point: as remarked above, translational invariance implies that $G_{\text{free}}(\vec{x}t,\vec{y}t'|s) = G_{\text{free}}(|\vec{x}-\vec{y}|,(t-t')|s)$. We are treating stochastic processes on a flat $d+2$-dimensional background (d-space dimensions plus real physical time t plus Schwinger proper time s).

Using the Taylor series representation

$$\frac{J_\ell(z)}{z^\ell} = \left(\frac{1}{2}\right)^\ell \sum_{n=0}^{+\infty} \frac{(-)^n(z/2)^{2n}}{n!\Gamma(\ell+n+1)}, \qquad (A10)$$

and integrating Eq. (A9) term-by-term, we find that (after making use of the time and space translational invariance of the Green function)

$$G_{\text{free}}(\vec{x}t,\vec{x}'t'|s) = \left(\frac{1}{4\pi s}\right)^{1/2} (s\nu^2)^{-d/4}$$

$$\times \exp\left[-\frac{(t-t')^2}{4s}\right]\frac{1}{2^{d+1}\pi^{d/2}}$$

$$\times \sum_{n=0}^{+\infty} \left(-\frac{1}{4}\right)^n \frac{\Gamma\left(\frac{n}{2}+\frac{d}{4}\right)}{n!\Gamma\left(\frac{d}{2}+n\right)} \left(\frac{|\vec{x}-\vec{x}'|^2}{\nu\sqrt{s}}\right)^n$$

$$\qquad\qquad (A11)$$

$$= A_d \exp\left[-\frac{(t-t')^2}{4s}\right] s^{-1/2-d/4} \sum_{n=0}^{+\infty} \left(\frac{-1}{4}\right)^n$$

$$\times C_{d,n} \frac{\Gamma\left(\frac{d}{2}\right)}{n!\Gamma\left(\frac{d}{2}+n\right)} \left(\frac{|\vec{x}-\vec{x}'|^2}{\nu\sqrt{s}}\right)^n, \qquad (A12)$$

where

$$A_d = \left(\frac{1}{4\pi}\right)^{1/2} \frac{\pi^{d/2}\Gamma\left(\frac{d}{4}\right)}{2(2\pi)^d\Gamma\left(\frac{d}{2}\right)} \left(\frac{1}{\nu^2}\right)^{d/4}, \quad \text{and}$$

$$C_{d,n} = \frac{\Gamma\left(\frac{d+2n}{4}\right)}{\Gamma\left(\frac{d}{4}\right)}. \tag{A13}$$

Special attention should be paid to the fact that this free Green function involves a series in *half-integer* powers of Schwinger proper time, $\sqrt{s}$, a feature that we use in choosing our ansatz for the full Green function.

APPENDIX B: HOMOGENEOUS FIELD CONFIGURATIONS

For constant (homogeneous and static) fields, $\phi = v_0$, the associated ''heat kernel'' can be solved for *exactly*. We simply present the final result and skip the details of a calculation that is an analog of that for $G_{\text{free}}(x,x'|s)$. The on-diagonal Green function $G_{\text{field}}^{(0)}(x,x|s)$ is given by

$$G_{\text{field}}^{(0)}(x,x|s) = A_d s^{-1/2-d/4} \exp[-s(V''[v_0]V[v_0]$$

$$+ V'[v_0]V'[v_0])]\sum_{\ell=0}^{+\infty} (\sqrt{s}\, 2\, V'[v_0])^\ell \frac{C_{d,\ell}}{\ell!}$$

$$= G_{\text{free}}(x,x|s)\exp[-s(V''[v_0]V[v_0]$$

$$+ V'[v_0]V'[v_0])]\sum_{\ell=0}^{+\infty} (\sqrt{s}\, 2\, V'([v_0])^\ell \frac{C_{d,\ell}}{\ell!},$$

$$\tag{B1}$$

where we have again made use of the definition $C_{d,\ell}$. We conclude that

$$\sum_{\ell=0}^{+\infty} [b_{\ell/2}]s^{\ell/2} = \exp[-s(V''[v_0]V[v_0]+V'[v_0]V'[v_0])]$$

$$\times \sum_{\ell=0}^{+\infty} (\sqrt{s}\, 2\, V'[v_0])^\ell \frac{C_{d,\ell}}{\ell!}. \tag{B2}$$

We now match the first fractional powers in s and obtain the Seeley-DeWitt coefficients for a constant field configuration v_0

$$[b_0] = 1, \tag{B3}$$

$$[b_{1/2}] = 2\, C_{d,1} V'[v_0], \tag{B4}$$

$$[b_1] = -V''[v_0]V[v_0]. \tag{B5}$$

The coefficients for $G_{\text{ghost}}^{(0)}$ can be immediately obtained from the previous Seeley-DeWitt coefficients by setting $V''[v_0] = 0$. That is

$$[a_0] = 1, \tag{B6}$$

$$[a_{1/2}] = 2\, C_{d,1} V'[v_0], \tag{B7}$$

$$[a_1] = \left(\frac{d}{2} - 1\right) V'[v_0] V'[v_0]. \tag{B8}$$

Finally, for the net Seeley-DeWitt coefficients, we can write

$$[c_0] = 0, \tag{B9}$$

$$[c_{1/2}] = 0, \tag{B10}$$

$$[c_1] = -V''[v_0]V[v_0]. \tag{B11}$$

These results are consistent with the net integrated Seeley-DeWitt coefficients and with the physical and ghost integrated Seeley-DeWitt coefficients presented in the body of the paper [Eqs. (59a)–(60d)].

APPENDIX C: THE FEYNMAN-HELLMAN FORMULA

We write the Feynman-Hellman formula in the form [30]

$$\frac{d}{d\epsilon}\exp(A + \epsilon B) = \int_0^1 d\ell\, \exp[\ell(A+\epsilon B)]B$$

$$\times \exp[(1-\ell)(A+\epsilon B)]. \tag{C1}$$

This equation is central to the computation of the Seeley-DeWitt coefficients presented in the paper. For instance, if we take the trace and then use the cyclic property, we can write

$$\frac{d}{d\epsilon}\text{Tr}[\exp(A+\epsilon B)] = \text{Tr}\{B\exp[(A+\epsilon B)]\}. \tag{C2}$$

We differentiate the previous equation to obtain

$$\frac{d^2}{d\epsilon^2}\text{Tr}[\exp(A+\epsilon B)] = \int_0^1 d\ell\, \text{Tr}\{B\exp[\ell(A+\epsilon B)]B$$

$$\times \exp[(1-\ell)(A+\epsilon B)]\}. \tag{C3}$$

We can conclude that

$$\text{Tr}[\exp(A+\epsilon B)] = \text{Tr}[\exp(A)] + \epsilon\, \text{Tr}[B\exp(A)]$$

$$+ \frac{\epsilon^2}{2}\int_0^1 d\ell\, \text{Tr}\{B\exp(\ell A)B$$

$$\times \exp[(1-\ell)A]\} + O(\epsilon^3). \tag{C4}$$

This perturbative expansion is the basis for extracting the first two *integrated* Seeley-DeWitt coefficients.

[1] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[2] D. Walgraef, *Spatio-Temporal Pattern Formation* (Springer, New York, 1997).

[3] G.I. Sivashinski, Acta Astron. **6**, 569 (1979).

[4] D. Hochberg, C. Molina-París, J. Pérez-Mercader, and M. Visser, Phys. Rev. E **60**, 6343 (1999), and references therein.

[5] D. Hochberg, C. Molina-París, J. Pérez-Mercader, and M. Visser, J. Stat. Phys. **99**, 903 (2000).

[6] J. Cardy, in *Proceedings of Mathematical Beauty of Physics*, Adv. Ser. in Math. Phys., Vol. 24, edited by J.B. Zuber (e-print cond-mat/9607163).

[7] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[8] S.A. Fulling, *Aspects of Quantum Field Theory in Curved Spacetime* (Cambridge University Press, Cambridge, England, 1989).

[9] A.A. Grib, S.G. Mamayev, and V.M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Friedmann Lab, St. Petersburg, Russia, 1994).

[10] B.S. DeWitt, in *General Relativity: An Einstein Centenary Survey*, edited by S.W Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[11] G. Gibbons, in *General Relativity: An Einstein Centenary Survey*, edited by S.W Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[12] See, e.g., *Quantum Field Theory Under the Influence of External Conditions*, edited by M. Bordag (Teubner, Leipzig, 1996).

[13] There is a wealth of excellent monographs and texts treating the subject of quantum field theory. A small and select sampling of some modern expositions may be found in Refs [14–17]. For more additional references, see the bibliography in Ref. [20].

[14] P. Ramond, *Field Theory: A Modern Primer* (Addison-Wesley, Massachusetts, 1981).

[15] J. Zinn–Justin, *Quantum Field Theory and Critical Phenomena*, 2nd ed. (Oxford University Press, Oxford, 1989).

[16] R.J. Rivers, *Path Integral Methods in Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1987).

[17] S. Weinberg, *The Quantum Theory of Fields I & II* (Cambridge University Press, Cambridge, England, 1996).

[18] Y. Fujimoto, L. O'Raifeartaigh, and G. Parravicini, Nucl. Phys. B **212**, 268 (1983).

[19] B. Gato, J. León, J. Pérez-Mercader, and M. Quirós, Nucl. Phys. B **253**, 285 (1985).

[20] D. Hochberg, C. Molina-París, J. Pérez-Mercader, and M. Visser, Int. J. Mod. Phys. A **14**, 1485 (1999).

[21] M. Doi, J. Phys. A **9**, 1465 (1976).

[22] L. Peliti, J. Phys. (France) Lett. **46**, 1469 (1985).

[23] P. Grassberger, F. Krause, and T. von der Twer, J. Phys. A **17**, L105 (1984).

[24] M.J. Howard and U.C. Tauber, J. Phys. A **30**, 7721 (1977).

[25] P.-A. Rey and J. Cardy, J. Phys. A **32**, 1585 (1999).

[26] H. Janssen, Z. Phys. B **42**, 151 (1981).

[27] J.D. Murray, *Mathematical Biology* (Springer-Verlag, Berlin, 1993).

[28] I.G. Avramidi and R. Schimming, in Ref. [12] and the references therein.

[29] Calculations of heat kernel coefficients for some specific fourth-order differential operators on curved backgrounds have been performed by V.P. Gusynin, Phys. Lett. B **255**, 233 (1989); Nucl. Phys. B **333**, 296 (1990).

[30] Assigning proper credit for this formula is extremely difficult: It is roughly equivalent to the Hellman-Feynman theorem of quantum mechanical forces, which was originally proven by P. Ehrenfest, Z. Phys. **45**, 455 (1927); and later discussed by Hellman (1937), and independently rediscovered by R. Feynman, Phys. Rev. **56**, 340 (1939); The version used here is equivalent to Eq. (2.181) of R. Feynman, *Statistical Mechanics* (Benjamin, Reading, Massachusetts, 1972), where Feynman simply called it ''perturbation theory.'' The formulas of Appendix C can also be found in the mathematical literature as several un-named lemmata on the way to proving the Baker-Campbell-Hausdorff formula [31].

[31] F. Hausdorff, ''Die symbolische Exponential Formel in der Gruppen Theorie,'' Leipziger Ber., **58**, 19 (1906); R. Gilmore,''Baker-Campbell-Hausdorff formulas,'' J. Math. Phys. **15**, 2090 (1974); J.A. Oteo, ''The Baker-Campbell-Hausdorff formula and nested commutator identities,'' *ibid.* **32**, 419 (1991); H. Kobayashi, N. Hatano, and M. Suzuki, ''Goldberg's theorem and the Baker-Campbell-Hausdorff formula,'' Physica A **250**, 535 (1998).

[32] S.K. Blau, M. Visser, and A. Wipf, Nucl. Phys. B **310**, 163 (1988).

[33] D. Hochberg, C. Molina–París, J. Pérez–Mercader, and M. Visser, Phys. Rev. Lett. **81**, 4802 (1998).

[34] J.B. Rundle, W. Klein, and S. Gross, Phys. Rev. Lett. **76**, 4285 (1996).